Perfectoid spaces

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$\boxed{\text{Last time}}$: (A, A^+) affinoid. We showed $\mathrm{Spa}\left(A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+\right) \overset{\text{homeo}}{\cong} R\left(\frac{f_1,\dots,f_n}{g}\right)$.

$\boxed{\text{Proof done modulo}}$:

① $R\left(\frac{f_1,\dots,f_n}{g}\right)$ quasi-compact

② $\underset{q\text{-cpt}}{X} \subseteq \mathrm{Spa}(A, A^+)$, $s \in \forall\, |\cdot| \in X: |s| \neq 0$

$\Rightarrow \exists\, 0 \in U$ open $\subseteq A$ s.t. $\forall\, |\cdot| \in X\ \forall a \in U: |a| \leq |s|$

<u>Idea</u>: $\forall\, |\cdot| \in X$: $|\cdot|$ continuous $\Rightarrow \{a \in A \mid |a| < |s|\} \supseteq I^{n_{|\cdot|}}$ ⇝ min'l power

$X_n = \{|\cdot| \in X \mid n_{|\cdot|} \leq n\}$, as $\bigcup X_n = X$: <u>enough</u> to show X_n is open

<u>On the side ①</u>: $A \underset{\text{open}}{\supseteq} A_0 \supseteq I$ ring of def., $a \in A_0 \Rightarrow I^n \cdot a^{\mathbb{Z}_{\geq 0}} \subseteq I^n \Rightarrow A_0 \subseteq A^0$

Fix generators $I = (a_1, \dots, a_r)_{A_0}$. $\overset{|A_0| \leq 1}{\Rightarrow}$ $\max |I^r| = \max\limits_{\sum k_i = r} \left|\prod_i a_i^{k_i}\right|$.

$\hookrightarrow r \geq n_{|\cdot|} \Leftrightarrow \forall \sum k_i = r: \left|\prod_i a_i^{k_i}\right| < |s|$

<u>On the side ②</u>: $I \subseteq A^{00}$

$\hookrightarrow X_r = \{|\cdot| \in X \mid \forall \sum k_i = r: |\prod_i a_i^{k_i}| < |s| \neq 0\} \supseteq \{|\cdot| \in X \mid \forall \sum k_i = r-1: |\prod_i a_i^{k_i}| \leq |s| \neq 0\}$ ↱ open, as rational open

<u>For ①, general picture</u>: $\underset{\substack{\text{all valuations,}\\ \text{same topology}}}{\mathrm{Spv}(A)} \underset{\text{closed}}{\supseteq} \mathrm{Cont}(A) \underset{\text{dense}}{\supseteq} \mathrm{Spa}(A, A^+)$

messy b/c of Supp $|\cdot|$'s can differ, $\mathrm{Spv}(A) \xrightarrow{\mathrm{Supp}} \mathrm{Spec}(A)$

details complicated, would take another 1-2 weeks...
Let's just show that one fiber of Supp is q-cpt.
<u>That is</u>:

$\boxed{\text{Prop}}$: K field $\Rightarrow \mathrm{Spv}(K)$ q-cpt.

$\boxed{\text{Pf}}$: $R\left(\frac{f_1,\dots,f_n}{g}\right) \overset{\text{def}}{=} \{|\cdot| \in \mathrm{Spv}(K) \mid \forall i: |f_i| \leq |g| \neq 0\} \Rightarrow g \neq 0 \Rightarrow \left|\frac{f_i}{g}\right| \leq 1$.

So instead of $R\left(\frac{f_1,\dots,f_n}{g}\right)$, we may use $U(a_1, \dots, a_n)$ (where $a_i = \frac{f_i}{g}$).

$U(a_1, \dots, a_n) := \{|\cdot| \in \mathrm{Spv} \mid a_i \in R_{|\cdot|}\}$

$X = \mathrm{Spv}(K)$.

<u>Want</u>: $\mathcal{A}$ is a collection of sets of the form $X \setminus U(a_1, \dots, a_n)$ s.t. no finitely many intersect in $\emptyset$ (*). Then $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$.

<u>Pf (want)</u>: By Zorn Lemma, we may assume that $\mathcal{A}$ max'l collection of closed sets of the above form satisfying (*)

<u>Claim 1</u>: $\bigcap_{F \in \mathcal{A}} F = \bigcap_{X \setminus U(a) \in \mathcal{A}} X \setminus U(a)$

<u>Pf (Claim 1)</u>: Need $\supseteq$. Take $|\cdot| \notin \bigcap_{F \in \mathcal{A}} F \Rightarrow$ fix $G \in \mathcal{A}$ s.t. $|\cdot| \notin G = X \setminus U(a_1, \dots, a_n)$

Note that $\bigcap_i U(a_i) = U(a_1,\dots,a_n) \Rightarrow \bigcup_i X\setminus U(a_i) = X\setminus U(a_1,\dots,a_n) = G$.

$\mathcal{A}$ max'l, $\mathcal{B}\subseteq\mathcal{A}$ finite $\Rightarrow \left(\bigcap_{H\in\mathcal{B}} H\right)\cap G \neq \emptyset \Rightarrow \exists j_{\mathcal{B}}: \left(\bigcap_{H\in\mathcal{B}} H\right)\cap(X\setminus U(a_j))\neq\emptyset$

↳ Let's assume that: $\forall i\ \exists \mathcal{B}_i \underset{\text{finite}}{\subseteq} \mathcal{A}: \left(\bigcap_{H\in\mathcal{B}_i} H\right)\cap(X\setminus U(a_i)) = \emptyset$.

Let $\mathcal{B} = \bigcup_i \mathcal{B}_i \Rightarrow \left(\bigcap_{H\in\mathcal{B}} H\right)\cap G\neq\emptyset \Rightarrow \exists j\ \left(\bigcap_{H\in\mathcal{B}} H\right)\cap(X\setminus U(a_j))\neq\emptyset$ ↯ □(Claim)

Final part: $\bigcap_{X\setminus U(a)\in\mathcal{A}} X\setminus U(a)\neq\emptyset$.

Pf (Final part): Note $X\setminus U(a) = \{|\cdot|\in \mathrm{Spv}(K) \mid R_{|\cdot|}\not\ni a\} = \{|\cdot|\in \mathrm{Spv}(K)\mid a^{-1}\in\mathfrak{m}_{|\cdot|}\}$

Let A = 'subring of K generated by all the a^{-1}'s s.t. $X\setminus U(a)\in\mathcal{A}$'

$\cup|$

I = 'ideal of A generated by all the a^{-1}'s ...'

$I\neq A$, because if $1=\sum_i^n b_i a_i^{-1}$, then we would have $\bigcap_{X\setminus U(a_i)\in\mathcal{A}} X\setminus U(a_i)=\emptyset$ ↯.

↳ $\exists\ \mathfrak{m}$ maximal ideal of A containing I $\Rightarrow$ $\exists$ valuation s.t. $|\cdot|\in \mathrm{Spv}(K)$: $|\mathfrak{m}|<1$

↳ $|\cdot|\in\bigcap_{X\setminus U(a)\in\mathcal{A}} X\setminus U(a)\neq\emptyset$. □(Final part) □(Want)

$\boxed{\text{Lemma}}$: A top. ring, $A_0\subseteq A$ subring. TFAE

(i) A f-adic w/ A_0 ring of def

(ii) $A_0\subseteq A$ open & bounded, and

condition on A: [$\exists\ U\subseteq A$ additive subgroup & $T\subseteq U$ finite s.t.
$\boxed{a}$ $\{U^n\}_n$ fund. system of nbhds of 0
$\boxed{b}$ $T\cdot U = U^2\subseteq U$]

$\boxed{\text{Pf}}$ (i) $\Rightarrow$ (ii): trivial

(ii) $\Rightarrow$ (i): A_0 open $\Rightarrow \exists\ r>0: U^r\subseteq A_0 \Rightarrow T^{(r)} := \{$r-fold products of elements of $T\}\subseteq A_0$.

Put $I = T^{(r)}\cdot A_0\subseteq A_0$.

① I^n open: $I^n = T^{(rn)}\cdot A_0 \supseteq T^{(rn)}\cdot U^r \overset{\boxed{b}}{=} U^{r(n+1)}$ open subgroup $\Rightarrow I^n$ open

② $V\subseteq A_0$ open $\underset{A_0 \text{ bounded}}{\Rightarrow} \exists\ U^n\cdot A_0\subseteq V \Rightarrow V\supseteq U^n\cdot A_0 \overset{\boxed{b}}{\supseteq} U^{nr}\cdot A_0 \supseteq I^n$ □

$\boxed{\text{Ex}}$: $\mathbb{Q}_p\langle x\rangle$. Usually one takes $\mathbb{Z}_p\langle x\rangle$ as ring of definition.

But by the above lemma, $\mathbb{Z}_p\langle x^2,x^3,px\rangle$ is also a ring of definition (as open and bounded).